CH :3 MOTION IN A PLANE

❖ Scalars and Vectors

On the basis of magnitude and direction, all the physical quantities are classified into two groups as Scalars and vectors.

- **Scalar Quantities**

These are the physical quantities which have only magnitude but no direction. It is specified completely by a single number, along with the proper unit.

e.g. Temperature, mass, length, time, work, etc.

- The rules for combining scalars follow simple rules of algebra. Scalars can be added, subtracted, multiplied and divided just as the ordinary numbers.
- The quantities with same units can be added or subtracted, but the quantities of different units can he multiplied or divided to make sense in scalars.

- **Vector Quantities**

These are the physical quantities which have both magnitudes and directions and obey the triangle/ parallelogram laws of addition and subtraction.

It is specified by giving its magnitude by a number and its direction. e.g. Displacement, acceleration, velocity, momentum, force, etc. A vector is represented by a bold face type and also by an arrow placed over a letter.

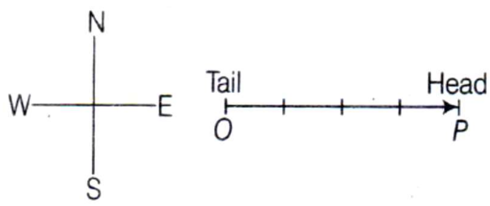
i.e. $\mathbf{F}, \mathbf{a}, \mathbf{b}$ or $\vec{F}, \vec{a}, \vec{b}$

The length of the line gives the magnitude and the arrowhead gives the direction.

e.g. Suppose a body has a Velocity 40 m/s due east. If 1 cm is chosen to represent a velocity of 10 m/s, then a line OP of 4 cm in length and drawn towards east with arrowhead at P will completely represent the velocity of the body (i.e. 40 m/s).

The point P is called head or terminal point and point O is called tail or initial point of the vector OP.

CH :3 MOTION IN A PLANE



Representation of vector OP

Vectors are classified into two types.

Polar Vectors

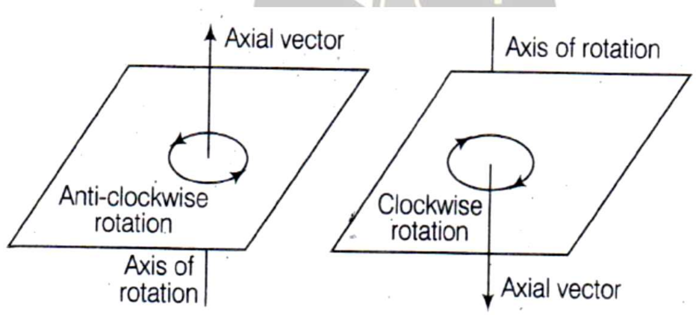
Vectors which have a starting point or a point of application are called polar vectors. e.g. Force, displacement, etc.

1. Axial Vectors

Vectors which represent the rotational effect and act along the axis of rotation are called axial vectors.

e.g. Angular velocity, angular momentum, torque, etc.

The axial vector will have its direction along its axis of rotation depending on its anti-clockwise or clockwise rotational effect.



- ✓ The magnitude of a vector is called modulus of that vector. For a vector A, it is represented by |v|= v.
- ✓ The physical quantities which have no specified direction and have different value in different directions are called tensors. e.g. Moment of inertia, stress, surface tension, pressure, etc.

❖ Important Definitions Related to Vectors

(i) Modulus of a vector

The magnitude of a vector is called modulus of that vector. For a vector A, it is represented by |A| or A

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(ii) Unit Vector

A vector having magnitude equal to unity but a specific direction is called a unit vector.

A Axial vector A unit vector of A is written as $\hat{A}$ and read as A cap. It is expressed as

$$\hat{\mathbf{A}} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\mathbf{A}}{A} = \frac{\text{Vector}}{\text{Magnitude of the vector}}$$

$$\boxed{\mathbf{A} = |\mathbf{A}|\hat{\mathbf{A}}}$$

Hence, any vector can be expressed as the magnitude times the unit vector along its own direction.

In cartesian coordinates, $\hat{\imath}, \hat{\jmath}$ and $\hat{k}$ are the unit vectors along X-axis, Y-axis and Z-axis respectively.

The magnitude of a unit vector is unity and has no unit or dimensions.

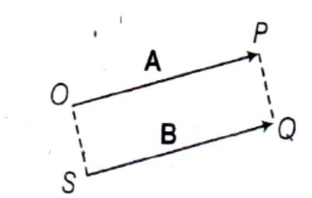
(iii) Null Vector

A vector with zero magnitude and having an arbitrary direction is called a null vector.

This vector is also known as zero vector and denoted by 0 (zero). e.g. The velocity vector of a stationary object, the acceleration vector of an object moving with uniform velocity.

(iv) Equal Vectors

Two vectors are said to be equal, if they have equal magnitude and same direction.



A and B are equal vectors

Consider two vectors A and B which are represented by two equal parallel lines drawn in e same magnitude and direction.

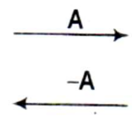
Thus, $\boldsymbol{OP} = \boldsymbol{SQ}$ or $\boldsymbol{A} = \boldsymbol{B}$

Vectors do not have fixed locations. When we displace a vector parallel to itself, then the vector does not change, such vectors are known as free vectors.

e.g. The velocity vector of a particle moving along a straight line is a free vector.

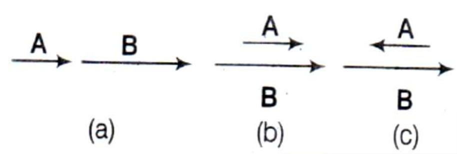
(v) Colinear Vectors

Two vectors are said to be the negative of each other, if their magnitudes are equal but directions are opposite. The negative vector of **A** is represented as **-A**.



(vi) Collinear Vectors

Two or more vectors are said to be collinear, when they act along the same lines or parallel lines. e.g. Tug of war.



If A and B are two collinear vectors, then they can be represented along a line in the same direction (Fig. (a)] or along the parallel lines in same direction [Fig.(b)] or along parallel lines in opposite direction [Fig.(c)].

(vii) Parallel Vectors

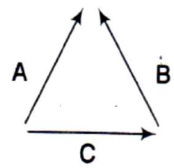
Two collinear vectors having the same directions are called parallel vectors. In this case, the angle between these two vectors will be zero.

(viii) Anti -parallel vector

Similarly, two collinear vectors having the opposite directions are called anti-parallel vectors. In this case, the angle between these two vectors will be 180°.

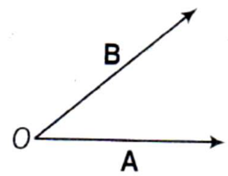
(ix) Co-planar Vectors

The vectors lying in the same plane are called co-planar vectors. Three vectors **A**, **B** and **C** are lying in the same plane of paper as shown in figure, hence they are co-planar vectors.



(x) Co-initial Vector

The vectors which have the same initial point are called co-initial vectors.



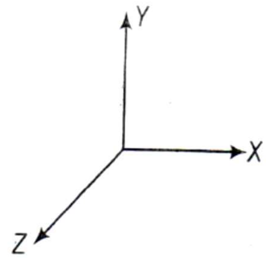
Co-initial vectors

Two vectors **A** and **B** have been drawn from the common initial point O. Therefore, **A** and **B** are called Co-initial vectors.

(xi) Orthogonal Unit Vector

If two- or three-unit vectors are perpendicular to each other, they are known as orthogonal unit vectors.

The unit vectors along X-axis, Y-axis and Z-axis are denoted by **i**, **j** and **k**. These are orthogonal unit vectors.



$$\hat{\mathbf{i}} = \frac{\mathbf{x}}{x} \Rightarrow \mathbf{x} = x\,\hat{\mathbf{i}}$$

$$\hat{\mathbf{j}} = \frac{\mathbf{y}}{y} \Rightarrow \mathbf{y} = y\,\hat{\mathbf{j}}$$

$$\hat{\mathbf{k}} = \frac{\mathbf{z}}{z} \Rightarrow \mathbf{z} = z\,\hat{\mathbf{k}}$$

(xii) Localised Vectors

Those vectors whose initial point is fixed are known as localised vectors., e.g. Position vector of a particle (initial point lies at the origin).

(xiii) Non-localised Vectors

Those vectors whose initial point is not fixed are known as non-localised. vectors. e.g. Velocity vector of a particle moving along a straight line.

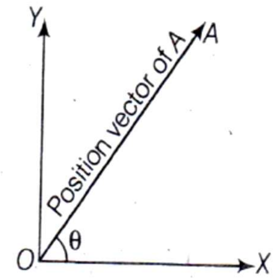
- ## Position and Displacement Vectors
  - ### Position Vector

A vector which gives position of an object with reference to the origin of a coordinate system is called position vector. It is represented by a symbol r.

Consider the motion of an object in XY-plane with origin at O. Suppose an object is at point A at any instant t, then OA s the position vector of the object at point A.

i.e. OA =r

CH :3 MOTION IN A PLANE



The position vector provides two informations such as

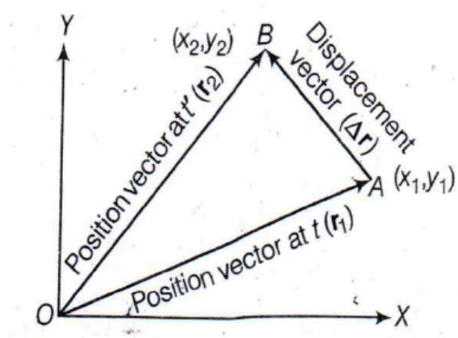
(i) It tells us about the minimum distance of an object from the origin O.
(ii) It tells us about the direction of the object w.r.t. origin.

○ Displacement Vector

The vector which tells how much and in which direction an object has changed its position in a given interval of time is called displacement vector.

Displacement vector is the straight line joining the initial and final positions and does not depend on the actual path undertaken by the object between the two positions.

Consider an object moving in the XY-plane. Suppose it is at point A at any instant t and at point B at any later instant t, then vector $\boldsymbol{AB}$ is the displacement vector of the object in time t to t'.



If the coordinates of points A and B are (x_1, y_1) and (x_2, y_2) respectively, then the position vector of the object at point A, $r_1 = x_1\hat{\imath} + y_1\hat{\jmath}$ by vector of the object at point: B, $r = x_2\hat{\imath} + y_2\hat{\jmath}$

∴ The displacement vector for B can be given as

$$\Delta r = \mathbf{r}_2 - \mathbf{r}_1$$

$$\text{Displacement vector, } \Delta r = (x_2 - x_1)\,\hat{\mathbf{i}} + (y_2 - y_1)\hat{\mathbf{j}}$$

- Magnitude of the displacement vector is given by

$$|\Delta \mathbf{r}| = \Delta r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The magnitude of displacement is either less or equal to the path length of an object between two points.

- Magnitude of vectors for three-dimensional is given by

$$\Delta \mathbf{r} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Multiplication of a Vector by a Real Number (or Scalar)

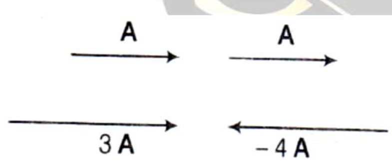
When we multiply a vector A by a real number, we get a new vector along the direction of vector **A**.

Its magnitude becomes λ times the magnitude of the given vector.

- Similarly, if we multiply vector A with a negative real number - λ, we get a vector whose magnitude is λ times the magnitude of vector A but direction is opposite to that of vector A.

  Hence, $\boxed{\lambda (A) = \lambda A}$ and $\boxed{-\lambda (A) = -\lambda A}$

  e.g. Consider a vector A is multiplied by a real number λ =3 or−4, we get 3A or −4A

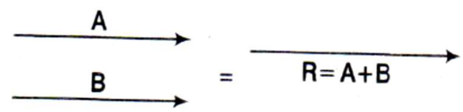


- If we multiply a constant velocity vector by time, we will get a displacement vector in the direction of velocity vector.

Resultant Vector

The resultant vector of two or more vectors is defined as the single vector which produces the same effect as two or more vectors (given vectors) combinedly produces. There are two cases

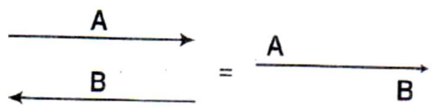
Case I When two vectors are acting in the same direction Consider the vectors A and B are acting in the same direction as shown below.



Then, the resultant of these two vectors is given by a vector having direction as same as that of A or B and the magnitude of the resultant vector will be equal to the sum of respective vectors. i.e. ($A + B$).

Thus, Resultant vector, $\mathbf{R} = \mathbf{A} + \mathbf{B}$

Case II When two vectors are acting in mutually opposite directions Consider the vectors A and B are acting in mutually opposite direction as shown below.



Then. the resultant of these two vectors is given by a vector having direction same as that of vector with larger magnitude. The magnitude of the resultant vector will be equal to $|A - B|$.

Thus, resultant vector, $R = A - B$

(i) IF $B > A$, then direction of R is along B.
(ii) If $A > B$, then direction of R is along A.

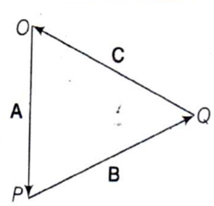
Conditions for Zero Resultant Vector

If three vectors acting on a point object at the same time are represented in magnitude and direction by the three sides of a triangle taken in the same order, their resultant is zero. The object is said to be in equilibrium.

Consider the three vectors A, B and C acting on an object at the same time represented by OP, PQ and QO, respectively.

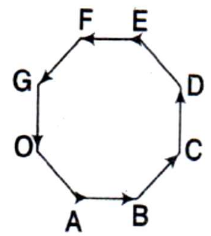
Then

$$\frac{A}{OP} = \frac{B}{PQ} = \frac{C}{QO}$$



Vectors A, B and C acting along OP, PQ and QO respectively

Similarly, if number of vectors acting GER an object at the same time are represented in magnitude and direction by the various sides of a cased polygon taken in the same order, their resultant vector is zero and the object will be in equilibrium.



Vectors represented by a closed polygon

Resultant vector, $R = OA + AB + BC + CD + DE + EF + FG + GO = 0$

Conditions for Equilibrium of an Object

(i) There is no linear motion of the object, i.e. the resultant force on the object is zero.
(ii) There is no rotational motion of the object, i.e. the torque due to forces on the object is zero.
(iii) There is minimum potential energy of the object for stable equilibrium.

Addition of Vectors (Graphical Method)

Two vectors can be added, if both of them are of same nature e.g. A displacement vector cannot be added to a force vector but can be added to displacement vector only.

Graphical method of addition of vectors helps us in visualising the vectors and the resultant vectors.

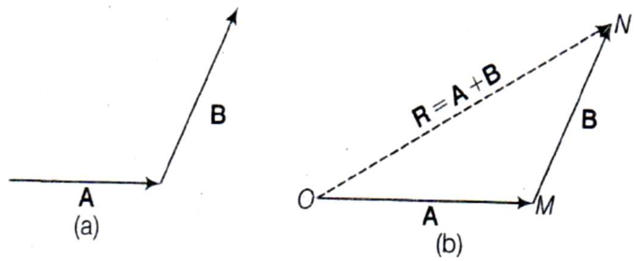
This method contains following laws

1. Triangle Law of Vector Addition

This law states that if two vectors can be represented both in magnitude and direction by the two sides of a triangle taken in the same order, then their resultant is represented completely, both in magnitude and direction, by the third side of the triangle taken in the opposite order.

Consider two vectors A and B that lie in a plane as shown in Fig. (a). Draw a vector OM equal and parallel to vector A as shown in Fig. (b).

From head of OM, draw a vector MN equal and parallel to vector B. Then, the resultant vector is given by ON which joins the tail of A and head of B.



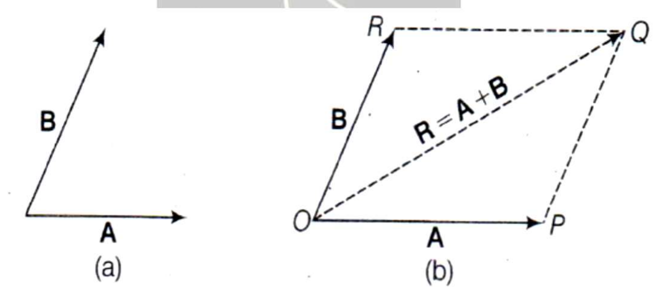
The resultant of A and B is $ON = OM + MN$

or Resultant vector, $R = A + B$

2. Parallelogram Law of Vector Addition

This law states that if two vectors acting on a particle at the same time are represented in magnitude and direction by two adjacent sides of a parallelogram drawn from a point, then their resultant vector is represented in magnitude and direction by the diagonal of the parallelogram drawn from the same point.

Consider two vectors A and B that lie in a plane as shown in Fig. (a). From a common point O, draw a vector OP equal and parallel to A and vector OR equal and parallel to B. Complete the parallelogram $OPQR$ as shown in Fig. (b), then the resultant vector is given by OQ.



According to parallelogram law of vector addition,

$$OQ = OP + OR$$

or Resultant vector, $R = A + B$

3. Polygon Law of Vector Addition

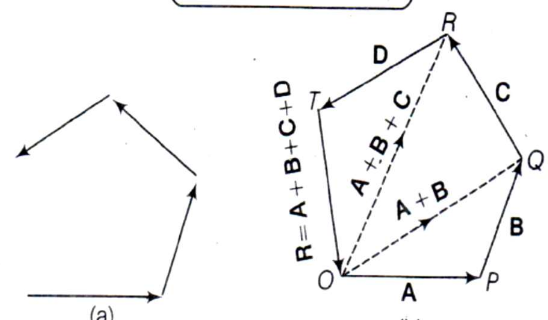
This law states that when the number of vectors are represented in both magnitude and direction by the sides of an open polygon taken in an order, then their resultant is represented in both magnitude and direction by the closing side of the polygon taken in opposite order.

Consider four vectors A, B, C and D be acting in different directions that lie in a plane as shown in Fig. (a)

Draw a vector OP parallel and equal to vector A. Move vectors B, C and D parallel to themselves, so that the tail of B touches the head of A, the tail of C touches the head of B and the tail of D touches the head of C as shown in Fig. (b).

According to the polygon law of vector addition, the closing side OT of the polygon taken in the reverse order represent the resultant R.

Thus, $$\boxed{R = A + B + C + D}$$



(a) (b)

- **Properties of Addition of Vectors**

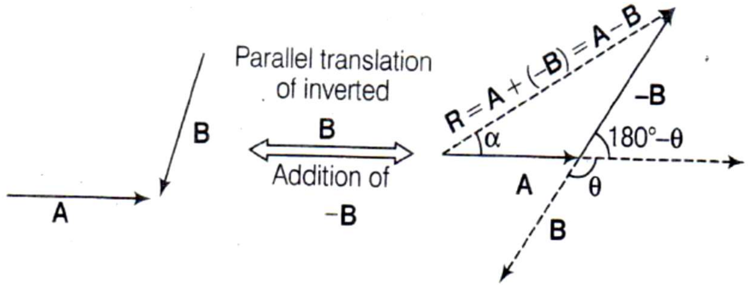
(i) It follows commutative law, i.e. $A + B = B + A$
(ii) It follows associative law, $(A + B) + C = A + (B + C)$
(iii) It obeys distributive law, λ(A + B) = λ A + λ B
(iv) $A + 0 = A$

- ## Subtraction of Two Vectors (Graphical Method)

if a vector B is to be subtracted from vector A, then we have to invert the vector B and then add it with vector A, according to laws of addition of two vectors.

Hence, the subtraction of vector B from a vector A is defined as the addition of vector $(-B)$ (i.e. negative of vector B) to vector A.

It is expressed as $R = A(-B) = A - B$



- **Properties of Subtraction of Vectors**

(i) Subtraction of vectors does not follow commutative law, i.e. $A - B \neq B - A$

(ii) It does not follow associative law, i.e. $A - (B - C) \neq (A - B) - C$

(iii) It follows distributive law, $\lambda(A - B) = \lambda A - \lambda B$.

- ## Resolution of Vectors in Plane (In Two Dimensions)

The process of splitting a single vector into two or more vectors in different directions which collectively produce the same effect as produced by the single vector alone is known as resolution of a vector.

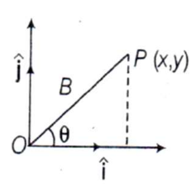
The various vectors into which a single vector is split are known as components of vectors. Resolution of a vector into two component vectors along the directions of two given vectors is unique.

To understand the resolution of a vector in the component vectors, let us discuss the vector as a combination of unit vectors.

Any vector r can be expressed as a l linear combination of two unit vectors $\hat{\imath}$ and $\hat{\jmath}$ at right angle,

i.e. $r = x\hat{\imath} + y\hat{\jmath}$

The vectors xi and yj are called the perpendicular components of r. The scalars x and y are called components or resolved parts of r in the directions of X-axis and Y-axis.



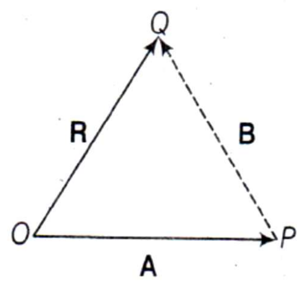
$\therefore$ $$\text{Resultant vector, } r = \sqrt{x^2 + y^2}$$

If θ is the inclination of r with X-axis, then

$$\text{Angle, } \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

- **Resolving a Vector into Two Component Vectors Along Given Directions**

Now, draw OQ to represent the resultant vector R in magnitude and direction. From point O, draw a line OP parallel to the vector A and from point P, draw a line PQ parallel to vector B.

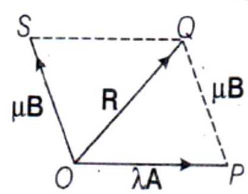


Then, these two lines intersect at point P as shown

From triangle law of vector addition, we have

$$OQ = OP + PQ$$

But OP and PQ are two component vectors of R in the direction of A and B, respectively. Let $OP = \lambda A$ and $PQ = \mu B$, λ where and u are two real numbers. This is also illustrated in the figure as shown. NA HB

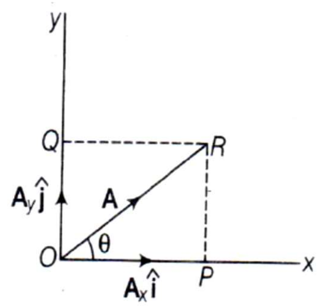


Now, the resultant vector becomes

$$\mathbf{R} = \lambda\mathbf{A} + \mu\mathbf{B}$$

- **Rectangular Components of a Vector in a Plane**

When a vector in a plane is split into two component vectors at right angle to each other, then the component vectors are called rectangular components of that vector. The resultant vector is given by



Rectangular components of A

$$\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}}$$

where, Magnitude of vector, $A = \sqrt{A_x^2 + A_y^2}$

We can also find the angle (∅) between them.

From $$\tan\theta = \left(\frac{A_y}{A_x}\right)$$

⇒ $$\text{Angle, } \theta = \tan^{-1}\left(\frac{A_y}{A_x}\right)$$

where, A_y and A_x are the split vectors component of A in the direction of $\hat{j}$ and $\hat{\imath}$, respectively.

- **Resolution of a Space Vector (In Three Dimensions)**

Similarly, we can resolve a general vector A into three components along X, Y and Z −axes in three dimensions (i.e. space).

Let α, β and γ be the angles between A and the X, Y and Z −axes, respectively as shown in figure.

Let $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ be the unit vectors, along X, Y and Z −axes, respectively.

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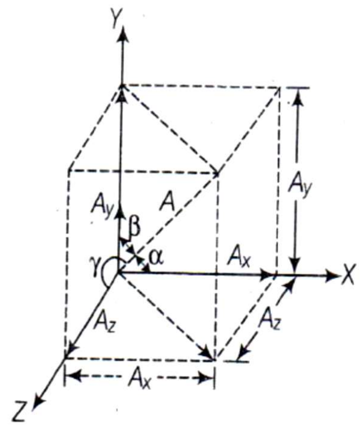


While resolving, we have

$$A_x = A\cos\alpha,\ A_y = A\cos\beta,\ A_z = A\cos\gamma$$

∴ Resultant vector, $A = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$

Magnitude of vector A, $A = \sqrt{A_x^2 + A_y^2 + A_z^2}$ ... (i)

Position vector r is given by $r = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

Remember that

$$\cos\alpha = \frac{A_x}{\sqrt{A_x^2 + A_y^2 + A_z^2}} = l$$

$$\cos\beta = \frac{A_y}{\sqrt{A_x^2 + A_y^2 + A_z^2}} = m$$

$$\cos\gamma = \frac{A_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}} = n$$

Here, I, m and n are known as direction cosines of A.

Putting the values of A_x, A_y and A_z in Eq. (i), we get

$$A^2 = A^2\cos^2\alpha + A^2\cos^2\beta + A^2\cos^2\gamma$$

$$A^2 = A^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma)$$

or $$\boxed{\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1}$$

It means, sum of. the squares of the direction cosines of a vector is always unity.

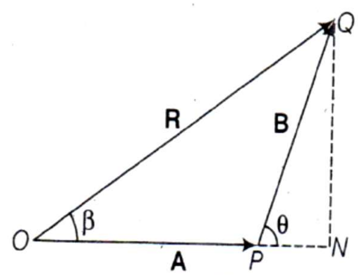
Note The angles α, β and γ are angles in space. They are between pairs of lines, which are not co-planar.

Addition of Vectors (Analytical Method)

Consider two vectors A and B inclined at an angle θ be acting on a particle at the same time.

Let them be represented in magnitude and direction by two sides OP and PO of ΔOPQ, taken in the same order.

Then, according to triangle law of vector addition, the resultant (R) is given by the closing side OQ, taken in opposite order.



Draw QN perpendicular to OP produced.

From ΔQNP, $\dfrac{PN}{PQ} = \cos\theta$

$\Rightarrow \quad PN = PQ\cos\theta = B\cos\theta \quad [\because PQ = B]$ ... (i)

and $\quad \dfrac{QN}{PQ} = \sin\theta$

$\Rightarrow \quad QN = PQ\sin\theta = B\sin\theta$ ... (ii)

In right angled ΔONQ we have

$$OQ^2 = QN^2 + NO^2 = QN^2 + (OP + PN)^2$$

or $\quad R^2 = (B\sin\theta)^2 + (A + B\cos\theta)^2$

$$= B^2\sin^2\theta + A^2 + B^2\cos^2\theta + 2\,AB\cos\theta$$

$$= A^2 + B^2 + 2A\,B\cos\theta$$

$\Rightarrow \quad \boxed{\text{Resultant, } R = \sqrt{A^2 + B^2 + 2AB\cos\theta}}$

This represents the magnitude of resultant vector R.

If the. resultant vector R makes an angle (β) with the direction of vector A, then from right angle ΔQNO,

$$\tan\beta = \frac{QN}{ON} = \frac{QN}{OP + PN} = \frac{B\sin\theta}{A + B\cos\theta}$$

or $$\text{Direction of resultant } \mathbf{R},\ \tan\beta = \frac{B\sin\theta}{A + B\cos\theta}$$

Regarding the Magnitude of R

(i) When $\theta = 0°$, then $R = A + B$ (maximum)
(ii) When $\theta = 90°$, then $R = \sqrt{A^2 + B^2}$.
(iii) When $\theta = 180°$, then $R = A - B$ (manimum)

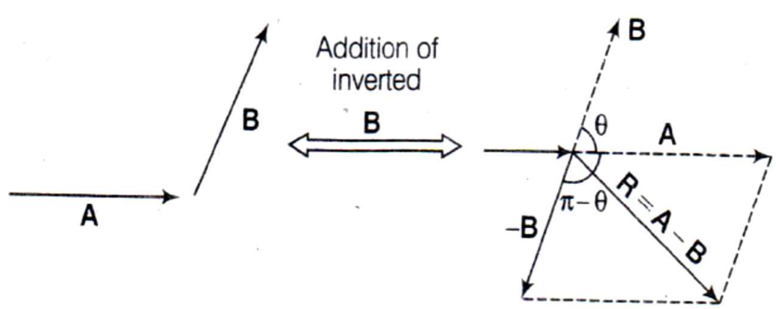
This can be extended to any number of vectors, if vectors a, b and c are c are given then

$$\mathbf{a} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}}$$
$$\mathbf{b} = b_x\hat{\mathbf{i}} + b_y\hat{\mathbf{j}} + b_z\hat{\mathbf{k}}$$
$$\mathbf{c} = c_x\hat{\mathbf{i}} + c_y\hat{\mathbf{j}} + c_z\hat{\mathbf{k}}$$

where, $\mathbf{r}_x = a_x + b_x + c_x$, $\mathbf{r}_y = a_y + b_y + c_y$
$\mathbf{r}_z = a_z + b_z + c_z$

- **Subtraction of Vectors (Analytical Method)**

There are two vectors A and B at an angle θ. If we have to subtract B from A, then first invert the vector B and then add with A as shown in figure.



The resultant vector is $\boldsymbol{R} = \boldsymbol{A} + (-\boldsymbol{B}) = \boldsymbol{A} - \boldsymbol{B}$

The magnitude of resultant in this case,

$$R = |\mathbf{R}| = \sqrt{A^2 + B^2 + 2AB\cos(\pi - \theta)}$$

$$\text{Resultant, } R = \sqrt{A^2 + B^2 - 2AB\cos\theta}$$

Here, θ = angle between A and B.

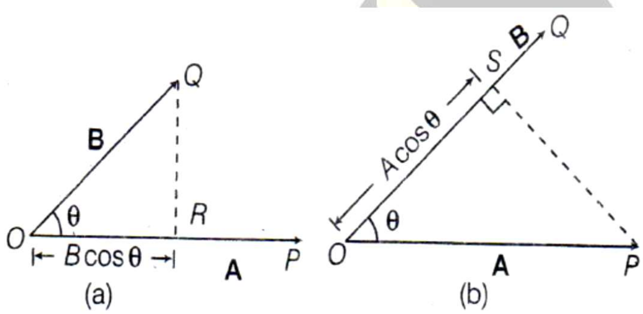
Regarding the magnitude of R

(i) When $\theta = 0°$, then $R = A - B$ (minimum)
(ii) When $\theta = 90°$, then $R = \sqrt{A^2 + B^2}$.
(iii) When $\theta = 180°$, then $R = A + B$ (maximum)

- **Dot Product or Scalar Product**

It is defined as the product of the magnitudes of vectors A and B and the cosine of the angle between them. It is represented by

$$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta$$



(a) B cos θ is the projection of B onto A.
(b) A cos θ is the projection of A onto B.

Case I When the two vectors are parallel, then θ =0°.

We have, $A \cdot B = AB$ cos 0°= AB

Case II When the two vectors are mutually perpendicular, then θ = 90°

We have, $A \cdot B$= AB cos 90°=0

Case III When the two vectors are anti-parallel, then θ =180°.

We have, $A \cdot B = AB$ cos 180° $= -AB$

- **Dot Product of Two Vectors in Terms of Their Components**

It is defined as the product of the magnitude of one vector and the magnitude of the component of other vector in the direction of first vector.

If $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$ and $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$,

then
$$\mathbf{a}\cdot\mathbf{b} = (a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}})\cdot(b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}})$$
$$= (a_1\cdot b_1)\hat{\mathbf{i}}\cdot\hat{\mathbf{i}} + (a_1\cdot b_2)\hat{\mathbf{i}}\cdot\hat{\mathbf{j}} + (a_1\cdot b_3)\hat{\mathbf{i}}\cdot\hat{\mathbf{k}}$$
$$+(a_2\cdot b_1)\hat{\mathbf{j}}\cdot\hat{\mathbf{i}} + (a_2\cdot b_2)\hat{\mathbf{j}}\cdot\hat{\mathbf{j}} + (a_2\cdot b_3)\hat{\mathbf{j}}\cdot\hat{\mathbf{k}}$$
$$+(a_3\cdot b_1)\hat{\mathbf{k}}\cdot\hat{\mathbf{i}} + (a_3\cdot b_2)\hat{\mathbf{k}}\cdot\hat{\mathbf{j}} + (a_3\cdot b_3)\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}.$$
$$= a_1\cdot b_1 + a_2\cdot b_2 + a_3\cdot b_3$$

where, $\hat{\mathbf{i}}\cdot\hat{\mathbf{i}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{k}}\cdot\hat{\mathbf{k}} = 1$

$\hat{\mathbf{i}}\cdot\hat{\mathbf{j}} + \hat{\mathbf{i}}\cdot\hat{\mathbf{k}} + \hat{\mathbf{j}}\cdot\hat{\mathbf{i}} + \hat{\mathbf{k}}\cdot\hat{\mathbf{i}} = 0$

Properties of Dot Product

(i) a · a =(a)2

(ii) a · b = b · a

(iii) a · (b + c) = a · b + a · c

(iv) (c · a) · b = c · (a · b)

(v) a · b = |a||b|cosθ

- ## Vector Product or Cross Product

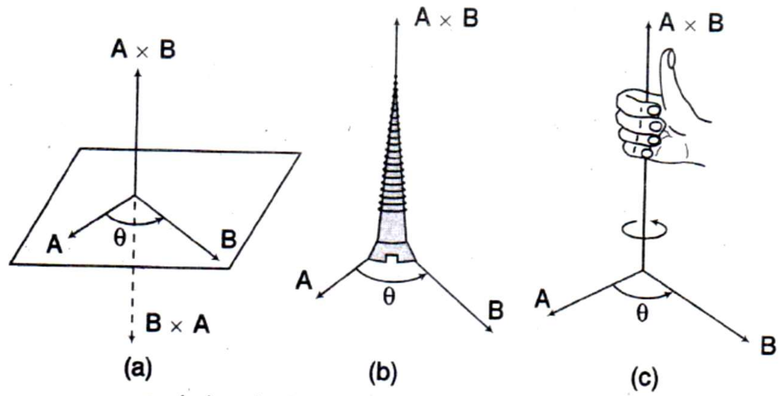
It is defined as the product of the magnitudes of vectors A and B and the sine of the angle 0 between them.

It is represented as

Cross product of vectors **A** and **B**, $\mathbf{A}\times\mathbf{B} = AB\sin\theta\,\hat{\mathbf{n}}$

where, n is a unit vector in the direction of $\hat{\imath}$

The following figures are representing the vector, product of vectors A and B (using right hand rule).



Right hand rules for direction of vector product

- **Cross Product of Two Vectors in Terms of Their Components**

If $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$ and $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$, then

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= (a_2b_3 - a_3b_2)\hat{\mathbf{i}} - (a_1b_3 - a_3b_1)\hat{\mathbf{j}} + (a_1b_2 - a_2b_1)\hat{\mathbf{k}}$$

where, $\hat{\mathbf{i}}\times\hat{\mathbf{i}} = \hat{\mathbf{j}}\times\hat{\mathbf{j}} = \hat{\mathbf{k}}\times\hat{\mathbf{k}} = 0$;

$\hat{\mathbf{i}}\times\hat{\mathbf{j}} = \hat{\mathbf{k}},\ \hat{\mathbf{j}}\times\hat{\mathbf{k}} = \hat{\mathbf{i}},\ \hat{\mathbf{k}}\times\hat{\mathbf{i}} = \hat{\mathbf{j}}$

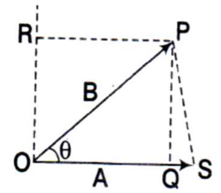
and $\hat{\mathbf{j}}\times\hat{\mathbf{i}} = -\hat{\mathbf{k}},\ \hat{\mathbf{k}}\times\hat{\mathbf{j}} = -\hat{\mathbf{i}},\ \hat{\mathbf{i}}\times\hat{\mathbf{k}} = -\hat{\mathbf{j}}.$

Case I Cross product of two parallel or anti-parallel vectors is zero.

Case II Cross product of two mutually perpendicular vectors is equal to product of the magnitude of two vectors

- If A and B are two sides of Δ, then area of

$$\Delta = \frac{1}{2}|\mathbf{A} \times \mathbf{B}|$$



Representation of vectors A and B

As, area of $\Delta = \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} OS \times PQ$

$$\left(\begin{array}{l} \because \ \sin\theta = \frac{PQ}{OP} \\ \Rightarrow \ PQ = OP\sin\theta = B\sin\theta \text{ and } OS = A \end{array} \right)$$

$$= \frac{1}{2} A \times B\sin\theta = \frac{1}{2} AB\sin\theta$$

$$= \frac{1}{2}|\mathbf{A} \times \mathbf{B}|$$

- **Properties of Cross Product**

(i) $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$

(ii) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$

(iii) $(\mathbf{a} \times \mathbf{b}) + (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \times \mathbf{c}) + (\mathbf{a} \times \mathbf{d}) + (\mathbf{b} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{d})$

(iv) $m\mathbf{a} \times \mathbf{b} = \mathbf{a} \times m\mathbf{b}$

(v) $(\mathbf{b} + \mathbf{c})\mathbf{a} = \mathbf{b} \times \mathbf{a} + \mathbf{c} \times \mathbf{a}$

(vi) $\mathbf{a} \times \mathbf{a} = 0$

(vii) $\mathbf{a} \times (\mathbf{b} - \mathbf{c}) = \mathbf{a} \times \mathbf{b} - \mathbf{a} \times \mathbf{c}$

(viii) $|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2 |\mathbf{b}|^2 - |\mathbf{a} \cdot \mathbf{b}|^2$

(ix) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c} \cdot \mathbf{a})\,\mathbf{b} - (\mathbf{b} \cdot \mathbf{a})\,\mathbf{c}$

- **Representation of Unit Vectors in a Circle**

Unit vectors along three axes of cartesian coordinate system (i.e. $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$) can be represented on a circle in such a way that, if we rotate our eyes in anti-clockwise direction, then product of two consecutive vectors will produce the third unit vector.

e.g. $\hat{i} \times \hat{j} = \hat{k}$ and $\hat{j} \times \hat{i} = -\hat{k}$

and similarly for other possibilities.

Scalar Product of Vectors

The scalar product of vectors produce pseudo scalars, such as volume, power, etc.

The vector product of two vectors produces a pseudo vector. It is also called axial vector. The direction of this vector is perpendicular to the plane containing the multiple vectors.

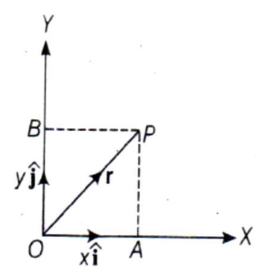
Motion in a Plane

Here, we will discuss how to describe motion of an object in two dimensions using vectors.

Position, Displacement and Velocity Vectors Position Vector

A vector that extends from a reference point to the point at which particle is located is called position vector.

Let r be the position vector of a particle P located in a plane with reference to the origin O in XY-plane as shown in figure.



$$\mathbf{OP} = \mathbf{OA} + \mathbf{OB}$$

$$\boxed{\text{Position vector, } \mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}}$$

In three dimensions, the position vector is represented

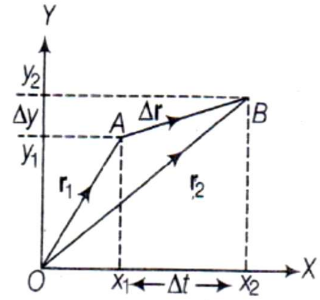
$$\boxed{r = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}}$$

CH :3 MOTION IN A PLANE

Displacement

Consider a particle moving in XY-plane with a uniform velocity v. Suppose O is the origin for measuring time and position of the particle.

Let the particle be at position A at time t_1 and at position B at time t_2, respectively. The position vectors are $OA = r_1$ and $OB = r_2$.



Then, the displacement of the particle in time interval ($t_2 — t_2$) is AB.

From triangle law of vector addition, we have

$$\mathbf{OA} + \mathbf{AB} = \mathbf{OB} \Rightarrow \mathbf{AB} = \mathbf{OB} - \mathbf{OA}$$

$$\boldsymbol{AB} = \boldsymbol{r_2} - \boldsymbol{r_1} \qquad \ldots \text{(i)}$$

If the position coordinates of the particle at points A and B are (x_1, y_1) and (x_2, y_2) respectively, then

$\therefore \qquad \mathbf{r}_1 = x_1\hat{\mathbf{i}} + y_1\hat{\mathbf{j}}$

and $\qquad \mathbf{r}_2 = x_2\hat{\mathbf{i}} + y_2\hat{\mathbf{j}}$

Substituting the values of $\mathbf{r}_1$ and $\mathbf{r}_2$ in Eq. (i), we have:

$$\mathbf{AB} = (x_2\hat{\mathbf{i}} + y_2\hat{\mathbf{j}}) - (x_1\hat{\mathbf{i}} + y_1\hat{\mathbf{j}})$$

$$\text{Displacement, } \mathbf{AB} = (x_2 - x_1)\,\hat{\mathbf{i}} + (y_2 - y_1)\,\hat{\mathbf{j}}$$

Similarly, in three dimensions, the displacement can be represented as

$$\Delta\mathbf{r} = (x_2 - x_1)\,\hat{\mathbf{i}} + (y_2 - y_1)\,\hat{\mathbf{j}} + (z_2 - z_1)\,\hat{\mathbf{k}}$$

Velocity

The rate of change of displacement of an object in particular direction is called its velocity.

It is of two types

Average Velocity

It is defined as the ratio of the displacement and the corresponding time interval.

Thus, average velocity $= \dfrac{\text{displacement}}{\text{time taken}}$

$$\text{Average velocity, } \mathbf{v}_{av} = \frac{\Delta \mathbf{r}}{\Delta t} = \frac{\mathbf{r}_2 - \mathbf{r}_1}{t_2 - t_1}$$

Velocity can be expressed in the component form as

$$\mathbf{v}_{av} = \frac{\Delta x}{\Delta t}\hat{\mathbf{i}} + \frac{\Delta y}{\Delta t}\hat{\mathbf{j}} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}}$$

where, v_x and v_y are the components of velocity along x −direction and y −direction, respectively.

The magnitude of v_{av} is given by $v_{av} = \sqrt{v_x^2 + v_y^2}$ and the direction of v_{av} is given by angle θ,

$$\tan\theta = \frac{v_y}{v_x}$$

$$\text{Direction of average velocity, } \theta = \tan^{-1}\left(\frac{v_y}{v_x}\right)$$

Instantaneous Velocity

The velocity at an instant of time (t) is known as instantaneous velocity.

The average velocity will become instantaneous, if Δt approaches to zero. The instantaneous velocity is expressed as

$$\text{Instantaneous velocity, } \mathbf{v}_i = \lim_{\Delta t \to 0}\frac{\Delta \mathbf{r}}{\Delta t} = \frac{d\mathbf{r}}{dt}$$

The limiting process can be easily understood with the help of figure.

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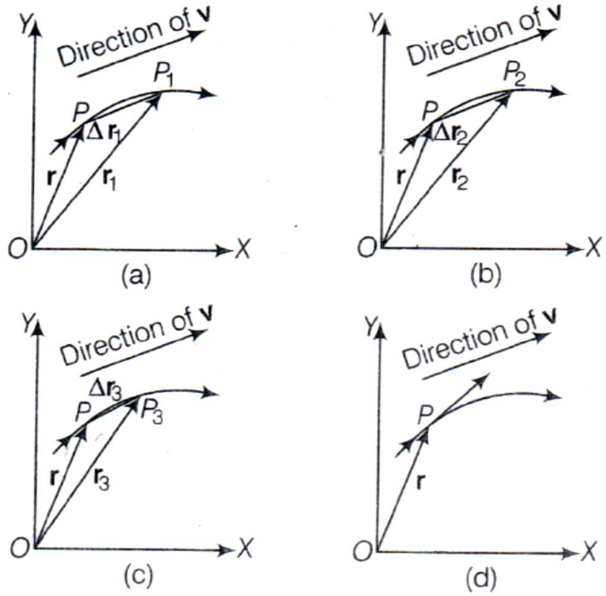


In the above figure, the curve represents the path of an object. The object is at point P on the path at time t.

P_1, P_2 and P_3 are the positions of the object after time intervals Δt_1, Δt_2 and Δt_3, where Δt_1, Δt_2 and Δt_3.

As the time interval Δt approaches zero, the average velocity approaches the velocity **v**. The direction of **v** is parallel to the line tangent to the path.

Note The direction of instantaneous velocity at any point on the path of an object is tangential to the path at that point and is in the direction of motion.

Acceleration

It is defined as the ratio of change in velocity and the corresponding time interval. It can be expressed as Acceleration, $\mathbf{a} = \frac{\text{Change in velocity}}{\text{Time taken}} = \frac{\Delta \mathbf{v}}{\Delta t}$

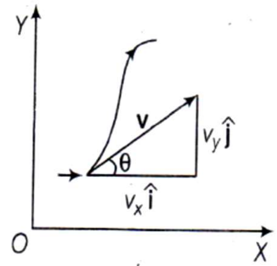
$$\text{Acceleration, } \mathbf{a} = \frac{\mathbf{v}_2 - \mathbf{v}_1}{t_2 - t_1}$$

Acceleration is also of two types

Average acceleration

It is defined as the change in velocity (Δv) divided by the corresponding time interval ((Δv).

It can be expressed as



Components of velocity

$$\text{Average acceleration, } \mathbf{a}_{av} = \frac{\Delta v}{\Delta t} = \frac{\Delta v_x \hat{\mathbf{i}} + \Delta v_y \hat{\mathbf{j}}}{\Delta t}$$

$$= \frac{\Delta v_x}{\Delta t}\hat{\mathbf{i}} + \frac{\Delta v_y}{\Delta t}\hat{\mathbf{j}}$$

$$\boxed{\text{Average acceleration, } \mathbf{a}_{av} = a_x \hat{\mathbf{i}} + a_y \hat{\mathbf{j}}}$$

which is expressed in component form.

In terms of x and y, a_x and a_y can be expressed as

$$a_x = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2} \quad \text{and} \quad a_y = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d^2y}{dt^2}$$

Instantaneous Acceleration

It is defined as the limiting value of the average acceleration as the time interval approaches to zero.

It can be expressed as

$$\mathbf{a}_i = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{v}}{\Delta t} = \frac{d\mathbf{v}}{dt}$$

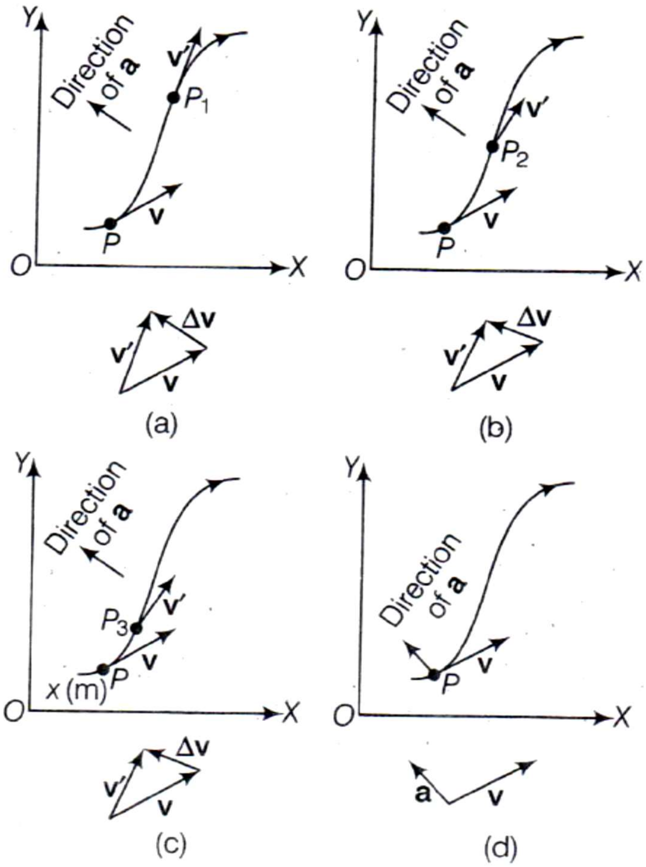
$$\boxed{\text{Instantaneous acceleration, } \mathbf{a}_i = a_x \hat{\mathbf{i}} + a_y \hat{\mathbf{j}}}$$

$$\text{where, } a_x = \frac{dv_x}{dt}, \; a_y = \frac{dv_y}{dt}$$

The magnitude of instantaneous acceleration is given by

$$a_i = \sqrt{a_x^2 + a_y^2}$$

The limiting process can be easily understood with the help of figure.



Limiting processes of instantaneous acceleration

The object is at point P at time t. P_1, P_2 and P_3 and represent the positions of the object after time intervals Δt_1, Δt_2 and Δt_3 respectively. The time interval at the different positions in such way that $\Delta t_1 > \Delta t_2 > \Delta t_3$.

The velocity vectors at points P, P_1, P_2 and P_3 are also shown in figures.

In each case of At, the change in velocity Δv is obtained by using triangle law of vector addition. The direction of the average acceleration is also shown as parallel to Δv.

The average acceleration for three intervals (a) Δt_1, (b) Δt_2 and (c) Δt_3, (d) ($\Delta t_1 > \Delta t_2 > \Delta t_3$).

In the limit $\Delta t \to 0$, the average acceleration becomes the instantaneous acceleration called simply acceleration.

From the given figures, it can be evaluated as the time interval Δt decreases from (a) to (d), the direction of Δt and hence that of a change.

In Fig. (d), the time interval $\Delta t \rightarrow 0$, hence, the average acceleration becomes the instantaneous acceleration having direction as shown in figure.

Note In two or three-dimensions, velocity and acceleration vectors can have any angle between 0° to 180, whereas in one-dimension, the velocity and acceleration of an object are always along the same straight line (may be in same direction or in opposite direction).

Motion in a Plane with Uniform Velocity

A body is said to be moving with uniform velocity, if it suffers equal displacements in equal intervals of time, however small. Consider an object moving with uniform velocity v in XY −plane. Let $r(0)$ and $r(t)$ be its position vectors at $t = 0$ and $t = t$ respectively,

then $$\mathbf{v} = \frac{\mathbf{r}(t) - \mathbf{r}(0)}{t - 0} \Rightarrow \mathbf{r}(t) = \mathbf{r}(0) + \mathbf{v}t \quad \dots \text{(i)}$$

In terms of rectangular coordinates, we get

$$\mathbf{v} = v_x \hat{\mathbf{i}} + v_y \hat{\mathbf{j}}, \quad v = \sqrt{v_x^2 + v_y^2}$$

$$\mathbf{r}(0) = x(0)\hat{\mathbf{i}} + y(0)\hat{\mathbf{j}} \quad \text{and} \quad \mathbf{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}}$$

On substituting these values in Eq. (i), we have

$$x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} = x(0)\hat{\mathbf{i}} + y(0)\hat{\mathbf{j}} + (v_x \hat{\mathbf{i}} + v_y \hat{\mathbf{j}})t$$

$$x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} = [x(0) + v_x t]\hat{\mathbf{i}} + [y(0) + v_y t]\hat{\mathbf{j}} \quad \dots \text{(ii)}$$

By equating the coefficients of $\hat{\imath}$ and $\hat{\jmath}$, we have

$$x(t) = x(0) + v_x t \quad \text{and} \quad y(t) = y(0) + v_y t$$

These two equations represent uniform motion along X −axis and Y −axis, respectively.

Eq. (ii) shows that the uniform motion in two- dimensions can be expressed as the sum of two uniform motions along two mutually perpendicular directions.

Motion in a Plane with Constant Acceleration

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CH :3 MOTION IN A PLANE

A body is said to be moving with uniform acceleration, if its velocity vector suffers the same change in the same interval of time, however small.

Let an object is moving in XY-plane and its acceleration a is constant. At time t = 0, the velocity of an object be v_0 (say) and v be the velocity at time t.

According to definition of average acceleration, we have

$$\mathbf{a} = \frac{\mathbf{v} - \mathbf{v}_0}{t - 0} = \frac{\mathbf{v} - \mathbf{v}_0}{t} \Rightarrow \boxed{\mathbf{v} = \mathbf{v}_0 + \mathbf{a}t}$$

In terms of rectangular components, we can express it as

$$v_x = v_{ox} + a_x t \text{ and } v_y = v_{oy} + a_y t$$

It can be concluded that each rectangular component of velocity of an object moving with uniform acceleration in a plane depends on time as, if it was the velocity vector of one-dimensional uniformly accelerated motion.

Now, we can also find the position vector (r). Let r_0 and r be the position vectors of the particle at time $t = 0$ and $t = t$ and their velocities at these instants be v_0 and v, then the average velocity is given by

$$\mathbf{v}_{av} = \frac{\mathbf{v}_0 + \mathbf{v}}{2}$$

Displacement is the product of average velocity and time interval. It is expressed as

$$\mathbf{r} - \mathbf{r}_0 = \left(\frac{\mathbf{v} + \mathbf{v}_0}{2}\right)t = \left[\frac{(\mathbf{v}_0 + \mathbf{a}t) + \mathbf{v}_0}{2}\right]t$$

$$\Rightarrow \qquad \mathbf{r} - \mathbf{r}_0 = \mathbf{v}_0 t + \frac{1}{2}\mathbf{a}t^2$$

$$\Rightarrow \qquad \boxed{\mathbf{r} = \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{a}t^2}$$

In terms of rectangular components, we have

$$x\hat{\mathbf{i}} + y\hat{\mathbf{j}} = x_0\hat{\mathbf{i}} + y_0\hat{\mathbf{j}} + (v_{0x}\hat{\mathbf{i}} + v_{0y}\hat{\mathbf{j}})t + \frac{1}{2}(a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}})t^2$$

Now, equating the coefficients of $\hat{\imath}$ and $\hat{\jmath}$,

$$x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2$$

$$y = y_0 + v_{0y}t + \frac{1}{2}a_y t^2$$

Note Motion in a plane (two-dimensional motion) can be treated as two separate simultaneous one-dimensional motions with constant acceleration along two perpendicular directions.

❖ Projectile Motion

Projectile motion is a form of motion in which an object or particle is thrown with some initial velocity near the earth's surface and it moves along a curved path under the action of gravity alone. The path followed by a projectile is called its trajectory.

An object that is in flight after being thrown is called projectile.

e.g.

(i) A tennis ball or a baseball in a flight.

(ii) A bullet fired from a rifle.

(iii) A body dropped from the window of a moving train.

(iv) A jet of water flowing from a hole near the bottom of water tank.

(v) A javelin thrown by an athlete.

• Assumptions before the Study of Projectile Motion

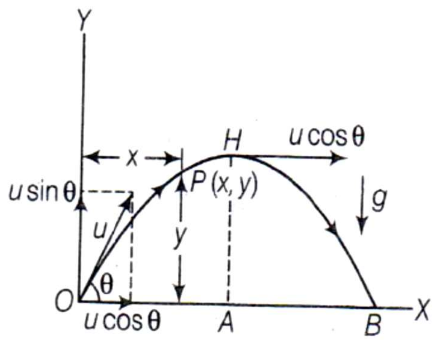
(i) There is no frictional resistance of air.
(ii) The effect due to rotation of earth and the curvature of the earth is negligible.
(iii) The acceleration due to gravity is constant both in magnitude and direction, at all points during the motion of projectile.

• Mathematical Analysis of Projectile Motion

Let OX be a horizontal line on the ground, OY be a vertical line perpendicular to ground and be the origin for XY −axes on a plane.

Suppose an object is projected from point O with velocity (μ), nmaking an angle (θ) with the horizontal direction OX, such that $x_0 = 0$ and $y_0 = 0$ when $t = 0$.

CH :3 MOTION IN A PLANE



While resolving velocity (μ) into two components, we get (a) μ cosθ along OX and (b) μ sinθ along OY

As the horizontal component of velocity ($\mu \cos \theta$) is constant throughout the motion, so there is a constant acceleration and hence, force is in the horizontal direction, if air resistance is assumed to be zero.

The vertical components of velocity (μ sin θ) decrease continuously with height, from O to H, due to downward force of gravity and becomes zero.

At point H, the object has only horizontal component velocity ($\mu \cos \theta$). It attains a maximum height at AH. OB is the maximum horizontal range.

Note The horizontal and vertical components of projectile motion was stated by Galileo.

- **Equation of Path of a Projectile**

Suppose at any time t_1, the object reaches at point $P(x, y)$.

So, $x =$ horizontal distance travelled by object in time t and y = vertical distance travelled by object in time t.

- **Motion Along Horizontal Direction (0X)**

The velocity of an object in horizontal direction, OX is constant, so the acceleration, in horizontal direction is zero.

∴ Position of the object at time t along horizontal direction is given by

$$x = x_0 + u_x t + \frac{1}{2} a_x t^2$$

But $x_0 = 0, u_x = u \cos\theta, a_x = 0$ and $t = t$

∴ $$x = u \cos\theta \; t$$

$\Rightarrow$ $$\text{Time, } t = \frac{x}{u\cos\theta} \quad \text{... (i)}$$

- **Motion Along Vertical Direction (0Y)**

The vertical velocity of the object is decreasing from O to H due to gravity, so acceleration a_y, is g.

∴ Position of the object at any time t along the vertical direction, i.e. OY is given by

$$y = y_0 + u_y t + \frac{1}{2} a_y t^2$$

But $y_0 = 0, u_y = u\sin\theta, a_y = -g$

and $t = t$

So, $$y = u\sin\theta\, t + \frac{1}{2}(-g)\, t^2$$

$$= u\sin\theta\, t - \frac{1}{2} g t^2 \quad \text{... (ii)}$$

Substituting the value of t from Eq. (i) in Eq. (ii), we get

$$y = u\sin\theta\left(\frac{x}{u\cos\theta}\right) - \frac{1}{2} g\left(\frac{x}{u\cos\theta}\right)^2$$

$$= x\tan\theta - \frac{1}{2} g\left(\frac{x}{u\cos\theta}\right)^2$$

$$\text{Total vertical distance, } y = x\tan\theta - \left(\frac{1}{2}\frac{g}{u^2\cos^2\theta}\right)x^2$$

This equation represents a parabola and is known as equation of trajectory of a projectile.

- **Time of Flight**

It is defined as the total time for which projectile is in flight, i.e. time during the motion of projectile from O to B. It is denoted by T.

Total time of flight consists of two parts such as

(a) Time taken by an object to go from point O to H. It is also known as time of ascent (t).
(b) Time taken by an object to go from point H to B. It is also known as time of descent (t).

Total time can be expressed as

$$T = t + t = 2t$$

$$\Rightarrow \qquad t = \frac{T}{2}$$

The vertical component of velocity of object becomes zero at the highest point H.

Let us consider vertical upward motion of an object from O to H, we have

$$u_y = u\sin\theta,\ a_y = -g,\ t = \frac{T}{2} \text{ and } v_y = 0$$

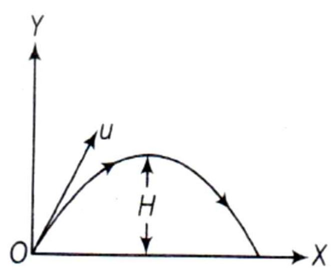
Since, $$v_y = u_y + a_y t \Rightarrow 0 = u\sin\theta - g\,\frac{T}{2}$$

$$\boxed{\text{Total time of flight, } T = \frac{2u\sin\theta}{g}}$$

For a projectile, time of ascent is to is equal time of descent.

- **Maximum Height of a Projectile**

It is defined as the maximum vertical height attained by an object above the point of projection during its flight. It is denoted by H.



Let us consider the vertical upward motion of the object from O to H, we have

$u_y = u\sin\theta,\ a_y = -g, y_0 = 0,\ y = H,\ t = \frac{T}{2} = \frac{u\sin\theta}{g}$

Using this relation, $y = y_0 + u_y t + \frac{1}{2} a_y t^2$

We have, $H = 0 + u\sin\theta\,\frac{u\sin\theta}{g} + \frac{1}{2}(-g)\left(\frac{u\sin\theta}{g}\right)^2$

$$= \frac{u^2}{g}\sin^2\theta - \frac{1}{2}\frac{u^2\sin^2\theta}{g}$$

$$\text{Maximum height, } H = \frac{u^2\sin^2\theta}{2g}$$

- **Horizontal Range of a Projectile**

The horizontal range of the projectile is defined as the horizontal distance covered by the projectile during its time of flight. It is denoted by R.

If the object has uniform velocity $\mu\cos\theta$(i.e. horizontal component) and the total time of iight T, then the horizontal range covered by the object,

$$\therefore \quad R = u\cos\theta \times T = u\cos\theta \times 2u\frac{\sin\theta}{g}$$

$$= \frac{u^2}{g} 2\sin\theta\cos\theta$$

$$\text{Horizontal range, } R = \frac{u^2\sin 2\theta}{g}$$

$[\because \sin 2\theta = 2\sin\theta\cos\theta]$

The horizontal range will be maximum, if

$\sin 2\theta = \text{maximum} = 1$

$\sin 2\theta = \sin 90^\circ$ or $\theta = 45^\circ$

$$\therefore \quad \text{Maximum horizontal range, } R_m = \frac{u^2}{g}$$

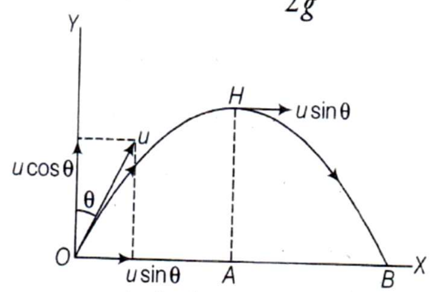
- **Projectile Fired at an Angle with the Vertical**

Let a particle be projected at an angle θ with vertical and its muzzle speed (i.e. speed of projection) is μ. The projectile has two components of its velocity at all the points during its motion.

The components are along X —axis (horizontal) and along Y —axis (vertical). Clearly, the angle made by the velocity of projectile at point of projection is $(90° - \theta)$ with the horizontal. In this case,

(i) Time of flight $= \dfrac{2u \sin(90° - \theta)}{g} = \dfrac{2u}{g}\cos\theta$

(ii) **Maximum height** $= \dfrac{u^2 \sin^2(90° - \theta)}{2g} = \dfrac{u^2 \cos^2\theta}{2g}$



Angle after projection

(iii) Horizontal range

$$= \frac{u^2}{g}\sin 2(90° - \theta)$$

$$= \frac{u^2}{g}\sin(180° - 2\theta) = \frac{u^2}{g}\sin 2\theta$$

(iv) Path of projectile

$$y = x\tan(90° - \theta) - \frac{1}{2}\frac{gx^2}{u^2\cos^2(90° - \theta)}$$

$$= x\cot\theta - \frac{gx^2}{2u^2\sin^2\theta}$$

(v) Velocity at any time, t

$$= \sqrt{[u\cos(90° - \theta)]^2 + [u\sin(90° - \theta) - gt]^2}$$

$$= \sqrt{u^2 + g^2t^2 - 2ugt\sin(90° - \theta)}$$

$$= \sqrt{u^2 + g^2t^2 - 2u\,gt\cos\theta}$$

This velocity makes an angle B with the horizontal direction, then

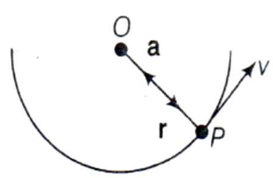
$$\tan\beta = \frac{u\sin(90° - \theta) - gt}{u\cos(90° - \theta)} = \frac{u\cos\theta - gt}{u\sin\theta}$$

- **Effect of Air Resistance on Projectile Motion**
  - As we have seen that in projectile motion, we assume that air resistance has no effect on its motion.
  - A projectile that traverses a parabolic path would slow during its idealised trajectory in the presence of air resistance. It will not hit the ground with the same speed with which it was projected.
  - In the absence of air resistance, it is only the y-component that undergoes a continuous change. However, in the presence of air resistance, both of x and y-components would get affected.

Uniform Circular Motion

When an object follows a circular path at a constant speed, the motion of the object is called uniform circular motion. The word uniform refers to the speed which is uniform (constant) throughout the motion. Although the speed does not vary, the particle is accelerating because the velocity changes its direction at every point on the circular track.

The following figure shows a particle P which moves along a circular track of radius r with a uniform speed μ.



Examples

(i) Motion of the tip of the second hand of a clock.
(ii) Motion of a point on the rim of a wheel rotating uniformly.

❖ Terms Related to Circular Motion

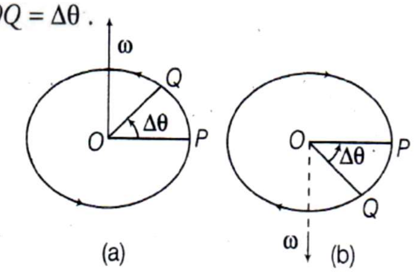
- **Angular Displacement**

It is defined as the angle traced out by the radius vector at the centre of the circular path in the given time. It is denoted by $\Delta\theta$ and expressed in radian. It is a dimensionless quantity. It is a vector quantity, direction is given by Right-hand rule.

- **Angular Velocity**

It is defined as the time rate of change of its angular position, denoted by ω and is measured in radian per second. Its dimensional formula is $[M^0L^0T^{-1}]$. It is a vector quantity.

Suppose a point object moving along a circular path, with centre (i.e. axis of rotation) at O. Let the object move from P to Q (as shown in the figure below) in a small time interval At, where $\angle POQ = \Delta\theta$.



Now, angular velocity, $\omega = \dfrac{\text{Angle traced}}{\text{Time taken}}$

$$\boxed{\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}}$$

- **Time Period**

It is defined as the time taken by a particle to complete one revolution along its circular path. It is denoted by T and is measured in second.

- **Frequency**

It is defined as the number of revolutions completed per unit time. It is denoted by f and is measured in Hz.

- **Angular Acceleration**

It is defined as the time rate of change of angular velocity of a particle. It is measured in radian per second square and has dimensions $[M^0L^0T^{-2}]$.

It is denoted by α, where $\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$

- **Relation between Time Period and Frequency**

Let f be the frequency of an object in circular motion, then the object will complete one revolution in $\frac{1}{f}$ second, which is known as time period (T).

$$\text{Time period, } T = \frac{1}{f}$$

Relation among Angular Velocity, Frequency and Time Period

Suppose a point object is illustrating a uniform circular motion with frequency (f) and time period (T). Then, the object completes one revolution and the angle traced at its axis of circular motion is 2 μ in radian.

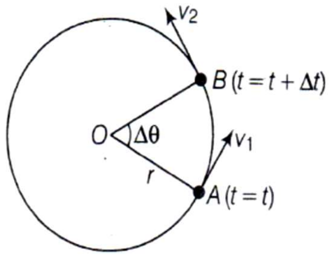
If time $t = T, \theta = 2\pi$

Thus, the angular velocity ω is given by

$$\text{Angular velocity, } \omega = \frac{\theta}{t} = \frac{2\pi}{T} = 2\pi f$$

- **Relation between Linear Velocity (v) and Angular Velocity (ω)**

Suppose the particle moving on circular track of radius r is showing angular displacement $\Delta\theta$ in Δt ie and in this time period, it covers a distance Δs long thẻ circular track, then



Representation of linear velocity and angular velocity

$$\omega = \frac{\Delta\theta}{\Delta t} \qquad \ldots(i)$$

and $$v = \frac{\Delta s}{\Delta t} \qquad \ldots(ii)$$

But $$\Delta s = r\,\Delta\theta \qquad \ldots(iii)$$

(Since, angle = arc/radius)

From above three equations, we get

$$\text{Linear velocity, } v = r\frac{\Delta\theta}{\Delta t} = r\,\omega$$

- **Centripetal Acceleration**

The acceleration associated with uniform circular motion is called a centripetal acceleration.

Consider a particle of mass (m) moving with a constant speed (v) and uniform angular velocity (w) on a circular path of radius (r) with centre at O.

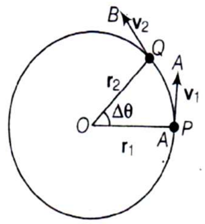
Suppose at any time t, the particle be at P, where $OP = r_1$ and at time $t + \Delta t$, the particle be at Q, where $OQ = r_2$, and $\angle POQ = \Delta\theta$ as shown in the figure. But $|r_1|=|r_2|=r$

Angular speed of the principle, $\omega = \frac{\Delta\theta}{\Delta t}$ … (i)

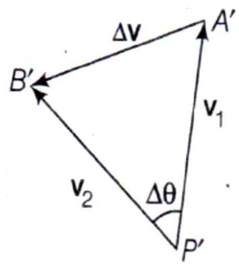
Let v_1 and v_2 be the velocity vectors of the particle at locations P and Q respectively. The magnitude and direction of v_1 and v_2 is represented by the tangent PA and QB.

Since the particle is moving with a uniform speed (v), the length of tangents at P and Q are equal.

i.e. $|PA| = |QB| = |\mathbf{v}|$



These two vectors have been separately shown in the following figures.



A triangle made of vectors

From triangle law of vectors, we have

$$P'A' + A'B' = P'B'$$

$$\Rightarrow \quad A'B' = P'B' - P'A' = v_2 - v_1 = \Delta v$$

If $\Delta t \rightarrow 0$, then A' lies close to B'. Then, $A'B'$ can be taken as an arc $A'B'$ of circle of radius $P'A' = |\text{v}|$

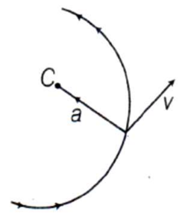
$$\therefore \quad \Delta\theta = \frac{A'B'}{P'A'} = \frac{|\Delta \mathbf{v}|}{|\mathbf{v}|}$$

From Eq. (i), we have $\omega \Delta t = \frac{|\Delta \mathbf{v}|}{|\mathbf{v}|} \Rightarrow \omega |\mathbf{v}| = \frac{|\Delta \mathbf{v}|}{\Delta t}$

$$\Rightarrow \quad \frac{|\Delta \mathbf{v}|}{\Delta t} = (\omega r)\,\omega = \omega^2 r \qquad [\because v = \omega r]$$

As $\Delta t \rightarrow 0$, then $\frac{|\Delta v|}{\Delta t}$ represents the magnitude of centripetal acceleration at P which is given by

$$|\mathbf{a}| = \frac{|\Delta \mathbf{v}|}{\Delta t} = \omega^2 r = \left[\frac{v}{r}\right]^2 r = \frac{v^2}{r}$$



$$\text{Centripetal acceleration, } a = \frac{v^2}{r}$$

It is towards the centre of circle.

- **Radius of Curvature**
  - Any curved path can be assumed to be a part of circular arc. Radius of curvature at a point is defined as the radius of that circular arc which fits at the particular point on the curve as shown in figure.

  - In the expression, $a_c = v^2/R$, the term R is known radius of curvature.

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CH :3 MOTION IN A PLANE

CH :3 MOTION IN A PLANE